8 Sets, Vectors and Functions

Bertrand Russell (1872–1970) tried to reduce all mathematics to formal logic. He showed that the idea of a set of all sets which are not members of themselves leads to contradictions. He wrote to Gottlieb Frege just as he was putting the finishing touches to a book that represented his life's work, pointing out that Frege's work was invalidated. Russell's elder brother, the second Earl Russell, showed great foresight in 1903 by queueing overnight outside the vehicle licensing office in London to have his car registered as A1.

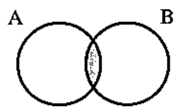
1 Use language, notation and Venn diagrams to describe sets

22 Use function notation to describe simple functions, and the notation $f^{-1}(x)$ to describe their inverses; form composite functions

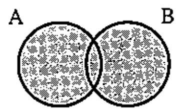
35 Add and subtract vectors; multiply a vector by a scalar; calculate the magnitude of a vector; represent vectors by directed line segments; express given vectors in terms of two coplanar vectors; use position vectors

8.1 Sets

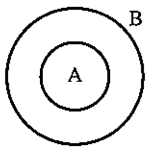
1. $\cap$ 'intersection'
$A \cap B$ is shaded.



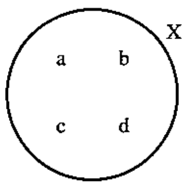
2. $\cup$ 'union'
$A \cup B$ is shaded.



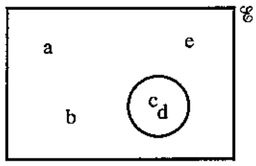
3. $\subset$ 'is a subset of'
$A \subset B$
[$B \not\subset A$ means 'B is *not* a subset of A']



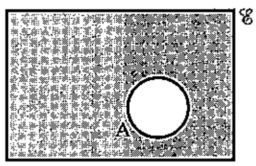
4. $\in$ 'is a member of'
'belongs to'
$b \in X$
[e $\notin$ X means 'e is not a member of set X']



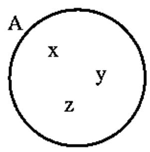
5. $\mathscr{E}$ 'universal set'
$\mathscr{E} = \{a, b, c, d, e\}$



6. A′ 'complement of'
'not in A'
A′ is shaded
($A \cup A' = \mathscr{E}$)



7. $n(A)$ 'the number of elements in set A'
$n(A) = 3$



8. $A = \{x : x \text{ is an integer}, 2 \leqslant x \leqslant 9\}$

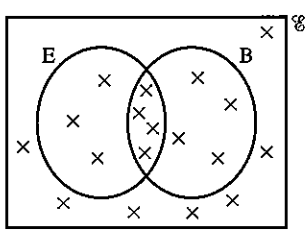
A is the $\boxed{\text{set of}}$ elements x $\boxed{\text{such that}}$ x is an integer and $2 \leqslant x \leqslant 9$.

The set A is {2, 3, 4, 5, 6, 7, 8, 9}.

9. $\varnothing$ or { } 'empty set'
(Note: $\varnothing \subset A$ for any set A)

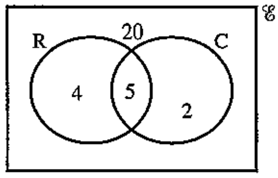
Exercise 1

1. In the Venn diagram,
$\mathscr{E}$ = {people in an hotel}
B = {people who like bacon}
E = {people who like eggs}



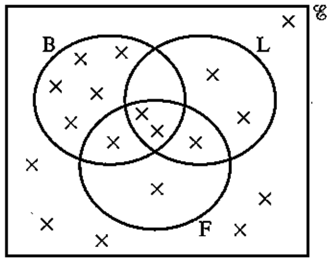
(a) How many people like bacon?
(b) How many people like eggs but not bacon?
(c) How many people like bacon and eggs?
(d) How many people are in the hotel?
(e) How many people like neither bacon nor eggs?

2. In the Venn diagram,
$\mathscr{E}$ = {boys in the fourth form}
R = {members of the rugby team}
C = {members of the cricket team}



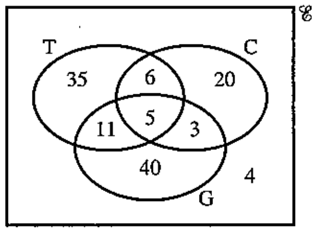
(a) How many are in the rugby team?
(b) How many are in both teams?
(c) How many are in the rugby team but not in the cricket team?
(d) How many are in neither team?
(e) How many are there in the fourth form?

3. In the Venn diagram,
$\mathscr{E}$ = {cars in a street}
B = {blue cars}
L = {cars with left-hand drive}
F = {cars with four doors}



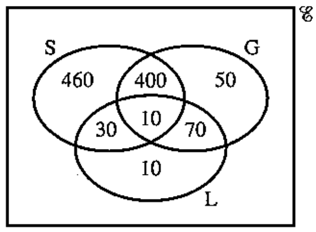
(a) How many cars are blue?
(b) How many blue cars have four doors?
(c) How many cars with left-hand drive have four doors?
(d) How many blue cars have left-hand drive?
(e) How many cars are in the street?
(f) How many blue cars with left-hand drive do not have four doors?

4. In the Venn diagram,
$\mathscr{E}$ = {houses in the street}
C = {houses with central heating}
T = {houses with a colour T.V.}
G = {houses with a garden}



(a) How many houses have gardens?
(b) How many houses have a colour T.V. and central heating?
(c) How many houses have a colour T.V. and central heating and a garden?
(d) How many houses have a garden but not a T.V. or central heating?
(e) How many houses have a T.V. and a garden but not central heating?
(f) How many houses are there in the street?

5. In the Venn diagram,
$\mathscr{E}$ = {children in a mixed school}
G = {girls in the school}
S = {children who can swim}
L = {children who are left-handed}

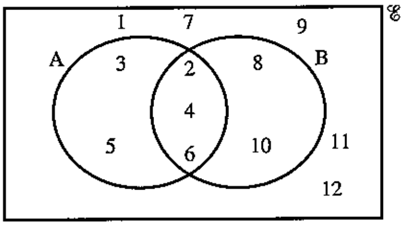


(a) How many left-handed children are there?
(b) How many girls cannot swim?
(c) How many boys can swim?
(d) How many girls are left-handed?
(e) How many boys are left-handed?
(f) How many left-handed girls can swim?
(g) How many boys are there in the school?

Example

$\mathscr{E} = \{1, 2, 3, \ldots, 12\}$, $A = \{2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8, 10\}$.

(a) $A \cup B = \{2, 3, 4, 5, 6, 8, 10\}$
(b) $A \cap B = \{2, 4, 6\}$
(c) $A' = \{1, 7, 8, 9, 10, 11, 12\}$
(d) $n(A \cup B) = 7$
(e) $B' \cap A = \{3, 5\}$



Exercise 2

In this exercise, be careful to use set notation only when the answer *is* a set.

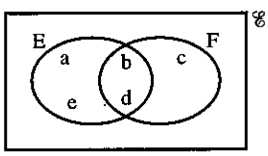
1. If $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $N = \{5, 7, 9, 11, 13\}$,
find:
(a) $M \cap N$ (b) $M \cup N$ (c) $n(N)$ (d) $n(M \cup N)$
State whether true or false:
(e) $5 \in M$ (f) $7 \in (M \cup N)$
(g) $N \subset M$ (h) $\{5, 6, 7\} \subset M$

2. If $A = \{2, 3, 5, 7\}$, $B = \{1, 2, 3, \ldots, 9\}$,
find:
(a) $A \cap B$ (b) $A \cup B$ (c) $n(A \cap B)$ (d) $\{1, 4\} \cap A$
State whether true or false:
(e) $A \in B$ (f) $A \subset B$ (g) $9 \subset B$ (h) $3 \in (A \cap B)$

3. If $X = \{1, 2, 3, \ldots, 10\}$, $Y = \{2, 4, 6, \ldots, 20\}$ and
$Z = \{x : x \text{ is an integer}, 15 \leqslant x \leqslant 25\}$,
find:
(a) $X \cap Y$ (b) $Y \cap Z$ (c) $X \cap Z$
(d) $n(X \cup Y)$ (e) $n(Z)$ (f) $n(X \cup Z)$
State whether true or false:
(g) $5 \in Y$ (h) $20 \in X$
(i) $n(X \cap Y) = 5$ (j) $\{15, 20, 25\} \subset Z$.

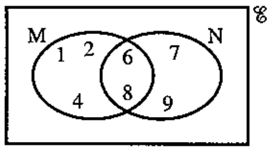
4. If $D = \{1, 3, 5\}$, $E = \{3, 4, 5\}$, $F = \{1, 5, 10\}$,
find:
(a) $D \cup E$ (b) $D \cap F$ (c) $n(E \cap F)$
(d) $(D \cup E) \cap F$ (e) $(D \cap E) \cup F$ (f) $n(D \cup F)$
State whether true or false:
(g) $D \subset (E \cup F)$ (h) $3 \in (E \cap F)$ (i) $4 \notin (D \cap E)$

5. Find:
(a) $n(E)$ (b) $n(F)$ (c) $E \cap F$
(d) $E \cup F$ (e) $n(E \cup F)$ (f) $n(E \cap F)$



6. Find:

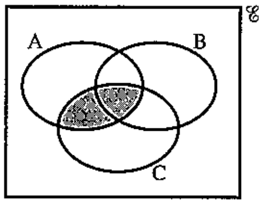
(a) $n(M \cap N)$ (b) $n(N)$ (c) $M \cup N$
(d) $M' \cap N$ (e) $N' \cap M$ (f) $(M \cap N)'$
(g) $M \cup N'$ (h) $N \cup M'$ (i) $M' \cup N'$



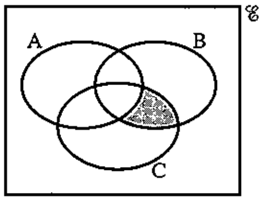
Example

On a Venn diagram, shade the regions:
(a) $A \cap C$ (b) $(B \cap C) \cap A'$
where A, B, C are interesting sets.

(a) $A \cap C$



(b) $(B \cap C) \cap A'$
[find $(B \cap C)$ first]



Exercise 3

1. Draw six diagrams similar to Figure 1 and shade the following sets:
(a) $A \cap B$ (b) $A \cup B$ (c) A'
(d) $A' \cap B$ (e) $B' \cap A$ (f) $(B \cup A)'$

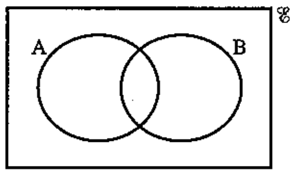


Figure 1

2. Draw four diagrams similar to Figure 2 and shade the following sets:
(a) $A \cap B$ (b) $A \cup B$ (c) $B' \cap A$ (d) $(B \cup A)'$

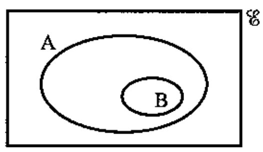


Figure 2

3. Draw four diagrams similar to Figure 3 and shade the following sets:
(a) $A \cup B$ (b) $A \cap B$ (c) $A \cap B'$ (d) $(B \cup A)'$

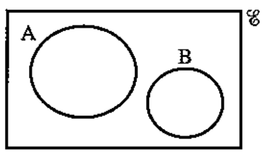


Figure 3

4. Draw eleven diagrams similar to Figure 4 and shade the following sets:

(a) $A \cap B$ (b) $A \cup C$ (c) $A \cap (B \cap C)$
(d) $(A \cup B) \cap C$ (e) $B \cap (A \cup C)$ (f) $A \cap B'$
(g) $A \cap (B \cup C)'$ (h) $(B \cup C) \cap A$ (i) $C' \cap (A \cap B)$
(j) $(A \cup C) \cup B'$ (k) $(A \cup C) \cap (B \cap C)$

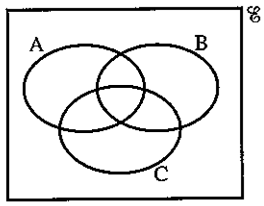


Figure 4

5. Draw nine diagrams similar to Figure 5 and shade the following sets:

(a) $(A \cup B) \cap C$ (b) $(A \cap B) \cup C$ (c) $(A \cup B) \cup C$
(d) $A \cap (B \cup C)$ (e) $A' \cap C$ (f) $C' \cap (A \cup B)$
(g) $(A \cap B) \cap C$ (h) $(A \cap C) \cup (B \cap C)$ (i) $(A \cup B \cup C)'$

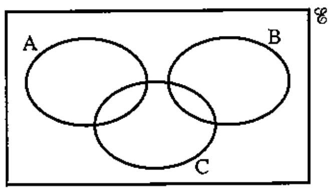
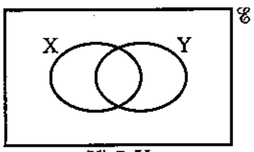
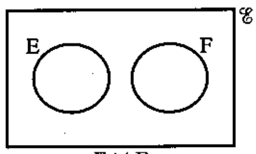


Figure 5

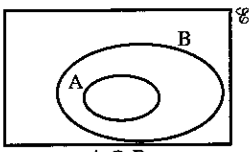
6. Copy each diagram and shade the region indicated.

(a)


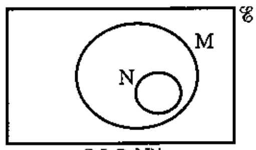
$X' \cap Y$

(b)


$E \cup F$

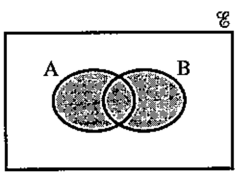
(c)


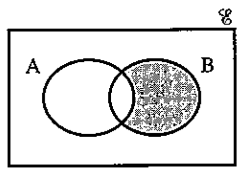
$A \cap B$

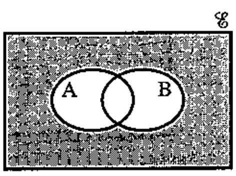
(d)


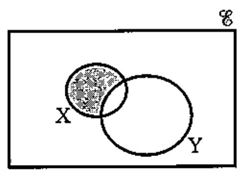
$(M \cap N)'$

7. Describe the region shaded.

(a)


(b)


(c)


(d)


8.2 Logical problems

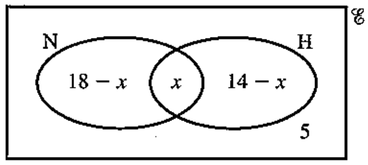
Example 1

In a form of 30 girls, 18 play netball and 14 play hockey, whilst 5 play neither.
Find the number who play both netball and hockey.

Let $\mathscr{E}$ = {girls in the form}
N = {girls who play netball}
H = girls who play hockey}

and x = the number of girls who play both netball and hockey

The number of girls in each portion of the universal set is shown in the Venn diagram.



$$\begin{aligned} \text{Since} \qquad n(\mathscr{E}) &= 30 \\ 18 - x + x + 14 - x + 5 &= 30 \\ 37 - x &= 30 \\ x &= 7 \end{aligned}$$

∴ Seven girls play both netball and hockey.

Example 2

If A = {sheep}
B = {sheep dogs}
C = {'intelligent' animals}
D = {animals which make good pets}

(a) Express the following sentences in set language:
(i) No sheep are 'intelligent' animals.
(ii) All sheep dogs make good pets.
(iii) Some sheep make good pets.

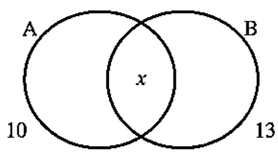
(b) Interpret the following statements:
(i) $B \subset C$
(ii) $B \cup C = D$

(a) (i) $A \cap C = \varnothing$
(ii) $B \subset D$
(iii) $A \cap D \neq \varnothing$

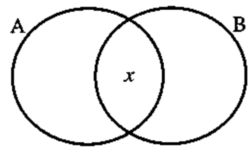
(b) (i) All sheep dogs are intelligent animals.
(ii) Animals which make good pets are either sheep dogs or 'intelligent' animals (or both).

Exercise 4

1. In the Venn diagram $n(A) = 10$, $n(B) = 13$, $n(A \cap B) = x$ and $n(A \cup B) = 18$.



(a) Write in terms of x the number of elements in A but not in B.
(b) Write in terms of x the number of elements in B but not in A.
(c) Add together the number of elements in the three parts of the diagram to obtain the equation $10 - x + x + 13 - x = 18$.
(d) Hence find the number of elements in both A and B.



2. In the Venn diagram $n(A) = 21$, $n(B) = 17$, $n(A \cap B) = x$ and $n(A \cup B) = 29$.
   (a) Write down in terms of x the number of elements in each part of the diagram.
   (b) Form an equation and hence find x.

3. The sets M and N intersect such that $n(M) = 31$, $n(N) = 18$ and $n(M \cup N) = 35$. How many elements are in both M and N?

4. The sets P and Q intersect such that $n(P) = 11$, $n(Q) = 29$ and $n(P \cup Q) = 37$. How many elements are in both P and Q?

5. The sets A and B intersect such that $n(A \cap B) = 7$, $n(A) = 20$ and $n(B) = 23$. Find $n(A \cup B)$.

6. Twenty boys in a form all play either football or basketball (or both). If thirteen play football and ten play basketball, how many play both sports?

7. Of the 53 staff at a school, 36 drink tea, 18 drink coffee and 10 drink neither tea nor coffee. How many drink both tea and coffee?

8. Of the 32 pupils in a class, 18 play golf, 16 play the piano and 7 play both. How many play neither?

9. Of the pupils in a class, 15 can spell 'parallel', 14 can spell 'Pythagoras', 5 can spell both words and 4 can spell neither. How many pupils are there in the class?

10. In a school, students must take at least one of these subjects: Maths, Physics or Chemistry. In a group of 50 students, 7 take all three subjects, 9 take Physics and Chemistry only, 8 take Maths and Physics only and 5 take Maths and Chemistry only. Of these 50 students, x take Maths only, x take Physics only and $x + 3$ take Chemistry only. Draw a Venn diagram, find x, and hence find the number taking Maths.

11. All of 60 different vitamin pills contain at least one of the vitamins A, B and C. Twelve have A only, 7 have B only, and 11 have C only. If 6 have all three vitamins and there are x having A and B only, B and C only and A and C only, how many pills contain vitamin A?

12. The IGCSE results of the 30 members of a Rugby squad were as follows: All 30 players passed at least two subjects, 18 players passed at least three subjects, and 3 players passed four subjects or more. Calculate:
    (a) how many passed exactly two subjects,
    (b) what fraction of the squad passed exactly three subjects.

13. In a group of 59 people, some are wearing hats, gloves or scarves (or a combination of these), 4 are wearing all three, 7 are wearing just a hat and gloves, 3 are wearing just gloves and a scarf and 9 are wearing just a hat and scarf. The number wearing only a hat or only gloves is x, and the number wearing only a scarf or none of the three items is $(x - 2)$. Find x and hence the number of people wearing a hat.

14. In a street of 150 houses, three different newspapers are delivered: T, G and M. Of these, 40 receive T, 35 receive G, and 60 receive M; 7 receive T and G, 10 receive G and M and 4 receive T and M; 34 receive no paper at all. How many receive all three?
Note: If '7 receive T and G', this information does not mean 7 receive T and G *only*.

15. If S = {Scottish men}, G = {good footballers}, express the following statements in words:
(a) $G \subset S$
(b) $G \cap S = \varnothing$
(c) $G \cap S \neq \varnothing$
(Ignore the truth or otherwise of the statements.)

16. Given that $\mathscr{E}$ = {pupils in a school}, B = {boys}, H = {hockey players}, F = {football players}, express the following in words:
(a) $F \subset B$ (b) $H \subset B'$ (c) $F \cap H \neq \varnothing$ (d) $B \cap H = \varnothing$
Express in set notation:
(e) No boys play football.
(f) All pupils play either football or hockey.

17. If $\mathscr{E}$ = {living creatures}, S = {spiders}, F = {animals that fly}, T = {animals which taste nice}, express in set notation:
(a) No spiders taste nice.
(b) All animals that fly taste nice.
(c) Some spiders can fly.
Express in words:
(d) $S \cup F \cup T = \mathscr{E}$ (e) $T \subset S$

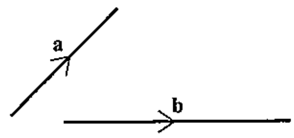
18. $\mathscr{E}$ = {tigers}, T = {tigers who believe in fairies}, X = {tigers who believe in Eskimos}, H = {tigers in hospital}.
Express in words:
(a) $T \subset X$ (b) $T \cup X = H$ (c) $H \cap X = \varnothing$
Express in set notation:
(d) All tigers in hospital believe in fairies.
(e) Some tigers believe in both fairies and Eskimos.

19. $\mathscr{E}$ = {school teachers}, P = {teachers called Peter}, B = {good bridge players}, W = {women teachers}. Express in words:
(a) $P \cap B = \varnothing$ (b) $P \cup B \cup W = \mathscr{E}$ (c) $P \cap W \neq \varnothing$
Express in set notation:
(d) Women teachers cannot play bridge well.
(e) All good bridge players are women called Peter.

8.3 Vectors

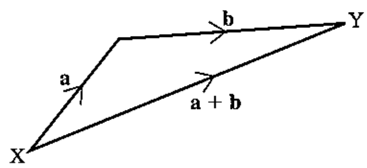
A vector quantity has both magnitude and direction. Problems involving forces, velocities and displacements are often made easier when vectors are used.

Addition of vectors

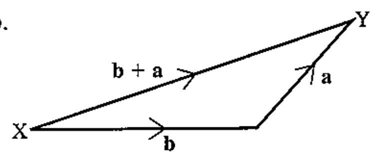


Vectors **a** and **b** represented by the line segments can be added using the parallelogram rule or the 'nose-to-tail' method.

The 'tail' of vector **b** is joined to the 'nose' of vector **a**.



Alternatively the tail of **a** can be joined to the 'nose' of vector **b**.

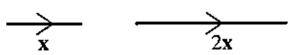


In both cases the vector $\overrightarrow{XY}$ has the same length and direction and therefore

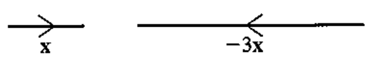
$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

Multiplication by a scalar

A scalar quantity has a magnitude but no direction (e.g. mass, volume, temperature). Ordinary numbers are scalars.
When vector **x** is multiplied by 2, the result is 2**x**.



When **x** is multiplied by −3 the result is −3**x**.



Note:
(1) The negative sign reverses the direction of the vector.
(2) The result of **a** − **b** is **a** + −**b**.
i.e. Subtracting **b** is equivalent to adding the negative of **b**.

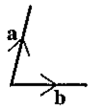
Example

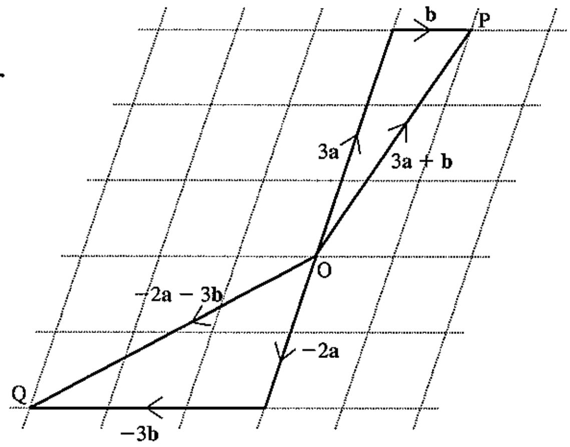
The diagram shows vectors **a** and **b**.

Find $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ such that

$\overrightarrow{OP} = 3\mathbf{a} + \mathbf{b}$

$\overrightarrow{OQ} = -2\mathbf{a} - 3\mathbf{b}$



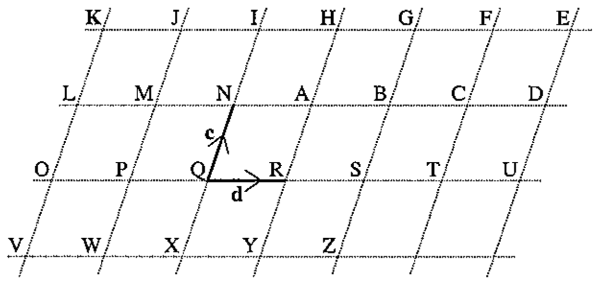


Exercise 5

In questions **1** to **26**, use the diagram below to describe the vectors given in terms of **c** and **d** where $\mathbf{c} = \overrightarrow{QN}$ and $\mathbf{d} = \overrightarrow{QR}$.

e.g. $\overrightarrow{QS} = 2\mathbf{d}$, $\overrightarrow{TD} = \mathbf{c} + \mathbf{d}$

1. $\overrightarrow{AB}$	**2.** $\overrightarrow{SG}$	**3.** $\overrightarrow{VK}$
4. $\overrightarrow{KH}$	**5.** $\overrightarrow{OT}$	**6.** $\overrightarrow{WJ}$
7. $\overrightarrow{FH}$	**8.** $\overrightarrow{FT}$	**9.** $\overrightarrow{KV}$
10. $\overrightarrow{NQ}$	**11.** $\overrightarrow{OM}$	**12.** $\overrightarrow{SD}$
13. $\overrightarrow{PI}$	**14.** $\overrightarrow{YG}$	**15.** $\overrightarrow{OI}$
16. $\overrightarrow{RE}$	**17.** $\overrightarrow{XM}$	**18.** $\overrightarrow{ZH}$
19. $\overrightarrow{MR}$	**20.** $\overrightarrow{KA}$	**21.** $\overrightarrow{RZ}$
22. $\overrightarrow{CR}$	**23.** $\overrightarrow{NV}$	**24.** $\overrightarrow{EV}$
25. $\overrightarrow{JS}$	**26.** $\overrightarrow{LE}$	



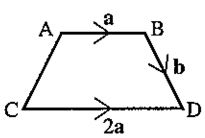
In questions **27** to **38**, use the same diagram above to find vectors for the following in terms of the capital letters, starting from Q each time.

e.g. $3\mathbf{d} = \overrightarrow{QT}$, $\mathbf{c} + \mathbf{d} = \overrightarrow{QA}$.

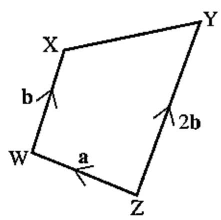
27. $2\mathbf{c}$	**28.** $4\mathbf{d}$	**29.** $2\mathbf{c} + \mathbf{d}$	**30.** $2\mathbf{d} + \mathbf{c}$
31. $3\mathbf{d} + 2\mathbf{c}$	**32.** $2\mathbf{c} - \mathbf{d}$	**33.** $-\mathbf{c} + 2\mathbf{d}$	**34.** $\mathbf{c} - 2\mathbf{d}$
35. $2\mathbf{c} + 4\mathbf{d}$	**36.** $-\mathbf{c}$	**37.** $-\mathbf{c} - \mathbf{d}$	**38.** $2\mathbf{c} - 2\mathbf{d}$

In questions **39** to **43**, write each vector in terms of **a** and/or **b**.

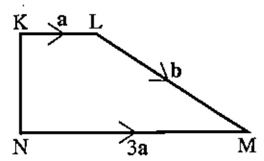
39. (a) $\overrightarrow{BA}$ (b) $\overrightarrow{AC}$
(c) $\overrightarrow{DB}$ (d) $\overrightarrow{AD}$



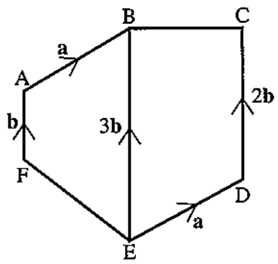
40. (a) $\overrightarrow{ZX}$ (b) $\overrightarrow{YW}$
(c) $\overrightarrow{XY}$ (d) $\overrightarrow{XZ}$



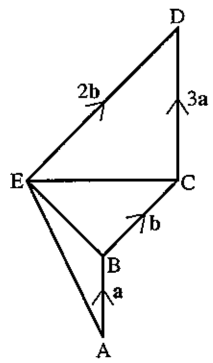
41. (a) $\overrightarrow{MK}$ (b) $\overrightarrow{NL}$ (c) $\overrightarrow{NK}$ (d) $\overrightarrow{KN}$



42. (a) $\overrightarrow{FE}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{FC}$ (d) $\overrightarrow{DA}$

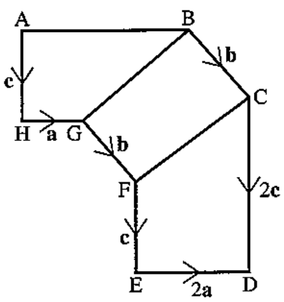


43. (a) $\overrightarrow{EC}$ (b) $\overrightarrow{BE}$ (c) $\overrightarrow{AE}$ (d) $\overrightarrow{EA}$

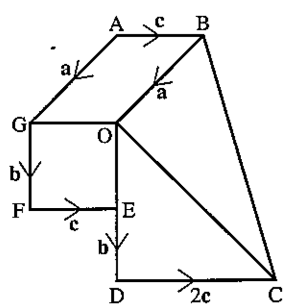


In questions **44** to **46**, write each vector in terms of **a**, **b** and **c**.

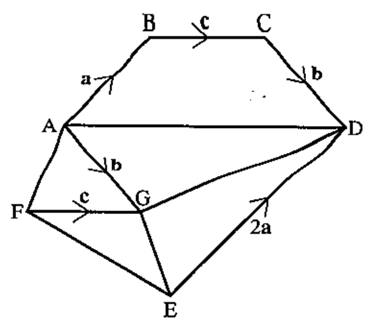
44. (a) $\overrightarrow{FC}$ (b) $\overrightarrow{GB}$ (c) $\overrightarrow{AB}$ (d) $\overrightarrow{HE}$ (e) $\overrightarrow{CA}$



45. (a) $\overrightarrow{OF}$ (b) $\overrightarrow{OC}$ (c) $\overrightarrow{BC}$ (d) $\overrightarrow{EB}$ (e) $\overrightarrow{FB}$



46. (a) $\overrightarrow{GD}$ (b) $\overrightarrow{GE}$ (c) $\overrightarrow{AD}$ (d) $\overrightarrow{AF}$ (e) $\overrightarrow{FE}$



Example

Using Figure 1, express each of the following vectors in terms of **a** and/or **b**.

(a) $\overrightarrow{AP}$ (b) $\overrightarrow{AB}$ (c) $\overrightarrow{OQ}$ (d) $\overrightarrow{PO}$ (e) $\overrightarrow{PQ}$
(f) $\overrightarrow{PN}$ (g) $\overrightarrow{ON}$ (h) $\overrightarrow{AN}$ (i) $\overrightarrow{BP}$ (j) $\overrightarrow{QA}$

OA = AP
BQ = 3OB
N is the mid-point of PQ
$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$

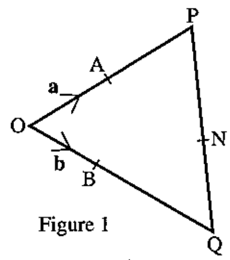


Figure 1

(a) $\overrightarrow{AP} = \mathbf{a}$

(b) $\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$

(c) $\overrightarrow{OQ} = 4\mathbf{b}$

(d) $\overrightarrow{PO} = -2\mathbf{a}$

(e) $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$
$= -2\mathbf{a} + 4\mathbf{b}$

(f) $\overrightarrow{PN} = \frac{1}{2}\overrightarrow{PQ}$
$= -\mathbf{a} + 2\mathbf{b}$

(g) $\overrightarrow{ON} = \overrightarrow{OP} + \overrightarrow{PN}$
$= 2\mathbf{a} + (-\mathbf{a} + 2\mathbf{b})$
$= \mathbf{a} + 2\mathbf{b}$

(h) $\overrightarrow{AN} = \overrightarrow{AP} + \overrightarrow{PN}$
$= \mathbf{a} + (-\mathbf{a} + 2\mathbf{b})$
$= 2\mathbf{b}$

(i) $\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP}$
$= -\mathbf{b} + 2\mathbf{a}$

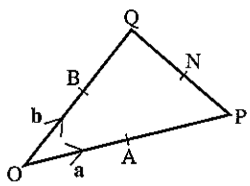
(j) $\overrightarrow{QA} = \overrightarrow{QO} + \overrightarrow{OA}$
$= -4\mathbf{b} + \mathbf{a}$

Exercise 6

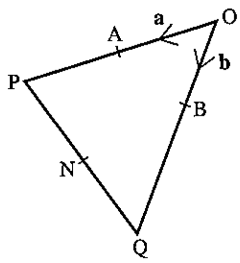
In questions **1** to **6**, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Copy each diagram and use the information given to express the following vectors in terms of **a** and/or **b**.

(a) $\overrightarrow{AP}$ (b) $\overrightarrow{AB}$ (c) $\overrightarrow{OQ}$ (d) $\overrightarrow{PO}$ (e) $\overrightarrow{PQ}$
(f) $\overrightarrow{PN}$ (g) $\overrightarrow{ON}$ (h) $\overrightarrow{AN}$ (i) $\overrightarrow{BP}$ (j) $\overrightarrow{QA}$

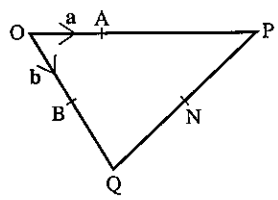
1. A, B and N are mid-points of OP, OQ and PQ respectively.



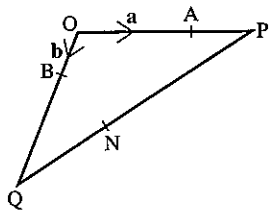
2. A and N are mid-points of OP and PQ and BQ = 2OB.



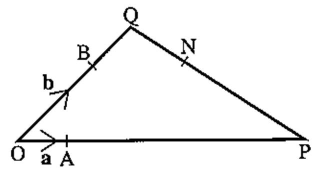
3. AP = 2OA, BQ = OB, PN = NQ.



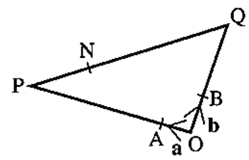
4. OA = 2AP, BQ = 3OB, PN = 2NQ.



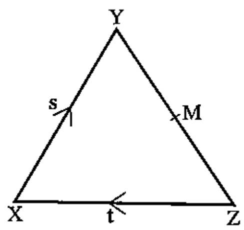
5. AP = 5OA, OB = 2BQ, NP = 2QN.



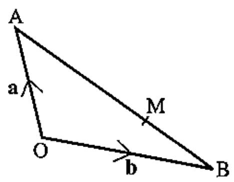
6. OA = $\frac{1}{5}$OP, OQ = 3OB, N is $\frac{1}{4}$ of the way along PQ.



7. In $\triangle$XYZ, the mid-point of YZ is M.
If $\overrightarrow{XY} = \mathbf{s}$ and $\overrightarrow{ZX} = \mathbf{t}$, find $\overrightarrow{XM}$ in terms of **s** and **t**.

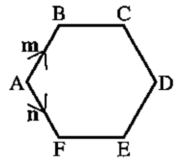


8. In $\triangle$AOB, AM : MB = 2 : 1. If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, find $\overrightarrow{OM}$ in terms of **a** and **b**.



9. O is any point in the plane of the square ABCD.
The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are **a**, **b** and **c** respectively. Find the vector $\overrightarrow{OD}$ in terms of **a**, **b** and **c**.

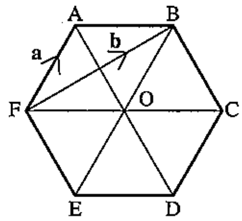
10. ABCDEF is a regular hexagon with $\overrightarrow{AB}$ representing the vector **m** $\overrightarrow{AF}$ representing the vector **n**. Find the vector representing $\overrightarrow{AD}$.



11. ABCDEF is a regular hexagon with centre O.
$\overrightarrow{FA} = \mathbf{a}$ and $\overrightarrow{FB} = \mathbf{b}$.

Express the following vectors in terms of **a** and/or **b**.

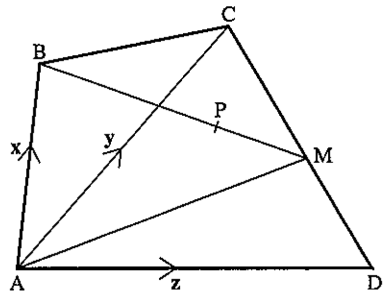
(a) $\overrightarrow{AB}$ (b) $\overrightarrow{FO}$ (c) $\overrightarrow{FC}$
(d) $\overrightarrow{BC}$ (e) $\overrightarrow{AO}$ (f) $\overrightarrow{FD}$



12. In the diagram, M is the mid-point of CD, BP : PM = 2 : 1, $\overrightarrow{AB} = \mathbf{x}$, $\overrightarrow{AC} = \mathbf{y}$ and $\overrightarrow{AD} = \mathbf{z}$.

Express the following vectors in terms of **x**, **y** and **z**.

(a) $\overrightarrow{DC}$ (b) $\overrightarrow{DM}$ (c) $\overrightarrow{AM}$
(d) $\overrightarrow{BM}$ (e) $\overrightarrow{BP}$ (f) $\overrightarrow{AP}$



8.4 Column vectors

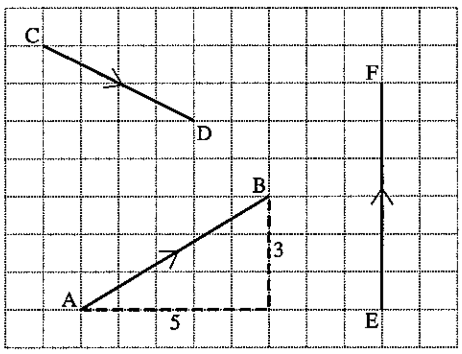
The vector $\overrightarrow{AB}$ may be written as a *column vector*.

$AB = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

The top number is the horizontal component of $\overrightarrow{AB}$ (i.e. 5) and the bottom number is the vertical component (i.e. 3).

Similarly $\overrightarrow{CD} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$\overrightarrow{EF} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$



Addition of vectors

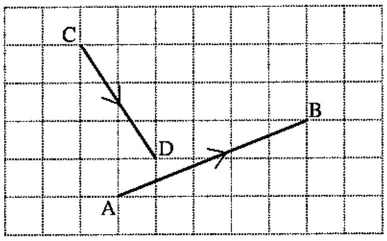


Figure 1

Suppose we wish to add vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in Figure 1.

First move $\overrightarrow{CD}$ so that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ join 'nose to tail' as in Figure 2. Remember that changing the *position* of a vector does not change the vector. A vector is described by its length and direction.

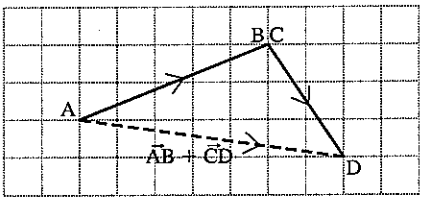


Figure 2

The broken line shows the result of adding $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

In column vectors,

$$\overrightarrow{AB} + \overrightarrow{CD} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

We see that the column vector for the broken line is $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$. So we perform addition with vectors by adding together the corresponding components of the vectors.

Subtraction of vectors

Figure 3 shows $\overrightarrow{AB} - \overrightarrow{CD}$.

To subtract vector $\overrightarrow{CD}$ from $\overrightarrow{AB}$ we *add* the *negative* of $\overrightarrow{CD}$ to $\overrightarrow{AB}$.

So $\overrightarrow{AB} - \overrightarrow{CD} = \overrightarrow{AB} + (-\overrightarrow{CD})$

In column vectors,

$$\overrightarrow{AB} + (-\overrightarrow{CD}) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

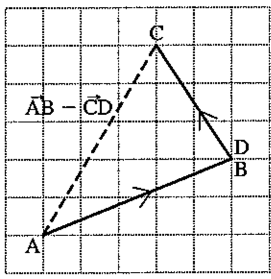


Figure 3

Multiplication by a scalar

If $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ then $2\mathbf{a} = 2\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}$.

Each component is multiplied by the number 2.

Parallel vectors

Vectors are parallel if they have the same direction. Both components of one vector must be in the same ratio to the corresponding components of the parallel vector.

e.g. $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ is parallel to $\begin{pmatrix} 6 \\ -10 \end{pmatrix}$,

because $\begin{pmatrix} 6 \\ -10 \end{pmatrix}$ may be written $2\begin{pmatrix} 3 \\ -5 \end{pmatrix}$.

In general the vector $k\begin{pmatrix} a \\ b \end{pmatrix}$ is parallel to $\begin{pmatrix} a \\ b \end{pmatrix}$.

Exercise 7

Questions **1** to **36** refer to the following vectors.

$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\mathbf{e} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ $\mathbf{f} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $\mathbf{g} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$ $\mathbf{h} = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$

Draw and label the following vectors on graph paper (take 1 cm to 1 unit).

1. $\mathbf{c}$ **2.** $\mathbf{f}$ **3.** $2\mathbf{b}$ **4.** $-\mathbf{a}$

5. $-\mathbf{g}$ **6.** $3\mathbf{a}$ **7.** $\frac{1}{2}\mathbf{e}$ **8.** $5\mathbf{d}$

9. $-\frac{1}{2}\mathbf{h}$ **10.** $\frac{3}{2}\mathbf{g}$ **11.** $\frac{1}{5}\mathbf{h}$ **12.** $-3\mathbf{b}$

Find the following vectors in component form.

13. $\mathbf{b} + \mathbf{h}$ **14.** $\mathbf{f} + \mathbf{g}$ **15.** $\mathbf{e} - \mathbf{b}$

16. $\mathbf{a} - \mathbf{d}$ **17.** $\mathbf{g} - \mathbf{h}$ **18.** $2\mathbf{a} + 3\mathbf{c}$

19. $3\mathbf{f} + 2\mathbf{d}$ **20.** $4\mathbf{g} - 2\mathbf{b}$ **21.** $5\mathbf{a} + \frac{1}{2}\mathbf{g}$

22. $\mathbf{a} + \mathbf{b} + \mathbf{c}$ **23.** $3\mathbf{f} - \mathbf{a} + \mathbf{c}$ **24.** $\mathbf{c} + 2\mathbf{d} + 3\mathbf{e}$

In each of the following, find $\mathbf{x}$ in component form.

25. $\mathbf{x} + \mathbf{b} = \mathbf{e}$ **26.** $\mathbf{x} + \mathbf{d} = \mathbf{a}$ **27.** $\mathbf{c} + \mathbf{x} = \mathbf{f}$

28. $\mathbf{x} - \mathbf{g} = \mathbf{h}$ **29.** $2\mathbf{x} + \mathbf{b} = \mathbf{g}$ **30.** $2\mathbf{x} - 3\mathbf{d} = \mathbf{g}$

31. $2\mathbf{b} = \mathbf{d} - \mathbf{x}$ **32.** $\mathbf{f} - \mathbf{g} = \mathbf{e} - \mathbf{x}$ **33.** $2\mathbf{x} + \mathbf{b} = \mathbf{x} + \mathbf{e}$

34. $3\mathbf{x} - \mathbf{b} = \mathbf{x} + \mathbf{h}$ **35.** $\mathbf{a} + \mathbf{b} + \mathbf{x} = \mathbf{b} + \mathbf{a}$ **36.** $2\mathbf{x} + \mathbf{e} = 0$ (zero vector)

37. (a) Draw and label each of the following vectors on graph paper.

$\mathbf{l} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$; $\mathbf{m} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$; $\mathbf{n} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$; $\mathbf{p} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; $\mathbf{q} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$;

$\mathbf{r} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$; $\mathbf{s} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$; $\mathbf{t} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$; $\mathbf{u} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$; $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

(b) Find four pairs of parallel vectors amongst the ten vectors.

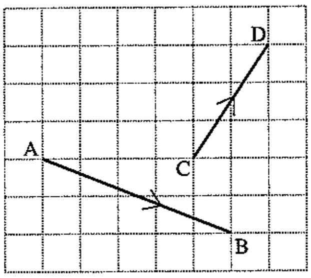
38. State whether 'true' or 'false'.

(a) $\begin{pmatrix}3\\-1\end{pmatrix}$ is parallel to $\begin{pmatrix}9\\-3\end{pmatrix}$ (b) $\begin{pmatrix}-2\\0\end{pmatrix}$ is parallel to $\begin{pmatrix}4\\0\end{pmatrix}$

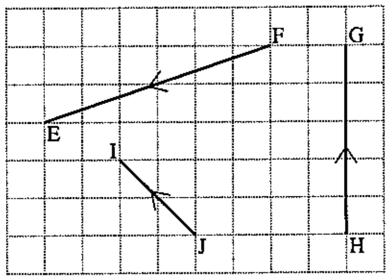
(c) $\begin{pmatrix}-1\\1\end{pmatrix}$ is parallel to $\begin{pmatrix}1\\-1\end{pmatrix}$ (d) $\begin{pmatrix}5\\-15\end{pmatrix} = 5\begin{pmatrix}1\\-3\end{pmatrix}$

(e) $\begin{pmatrix}4\\0\end{pmatrix}$ is parallel to $\begin{pmatrix}0\\6\end{pmatrix}$ (f) $\begin{pmatrix}3\\-1\end{pmatrix} + \begin{pmatrix}-4\\-2\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

39. (a) Draw a diagram to illustrate the vector addition $\overrightarrow{AB} + \overrightarrow{CD}$.
(b) Draw a diagram to illustrate $\overrightarrow{AB} - \overrightarrow{CD}$.



40. Draw separate diagrams to illustrate the following.
(a) $\overrightarrow{FE} + \overrightarrow{JI}$
(b) $\overrightarrow{HG} + \overrightarrow{FE}$
(c) $\overrightarrow{JI} - \overrightarrow{FE}$
(d) $\overrightarrow{HG} + \overrightarrow{JI}$

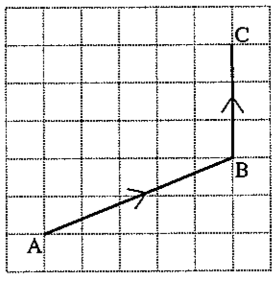


Exercise 8

1. If D has coordinates (7, 2) and E has coordinates (9, 0), find the column vector for $\overrightarrow{DE}$.

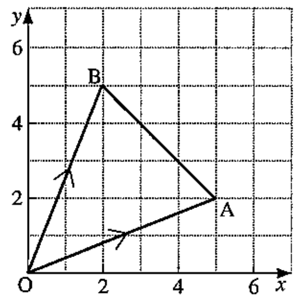
2. Find the column vector $\overrightarrow{XY}$ where X and Y have coordinates $(-1, 4)$ and $(5, 2)$ respectively.

3. In the diagram $\overrightarrow{AB}$ represents the vector $\begin{pmatrix}5\\2\end{pmatrix}$ and $\overrightarrow{BC}$ represents the vector $\begin{pmatrix}0\\3\end{pmatrix}$.
(a) Copy the diagram and mark point D such that ABCD is a parallelogram.
(b) Write $\overrightarrow{AD}$ and $\overrightarrow{CA}$ as column vectors.



4. (a) On squared paper draw $\overrightarrow{AB} = \begin{pmatrix}3\\-2\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}4\\2\end{pmatrix}$ and mark point D such that ABCD is a parallelogram.
(b) Write $\overrightarrow{AD}$ and $\overrightarrow{CA}$ as column vectors.

5. Copy the diagram in which $\overrightarrow{OA} = \begin{pmatrix}5\\2\end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix}2\\5\end{pmatrix}$.



M is the mid-point of AB. Express the following as column vectors:
(a) $\overrightarrow{BA}$ (b) $\overrightarrow{BM}$ (c) $\overrightarrow{OM}$ (use $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$)
Hence write down the coordinates of M.

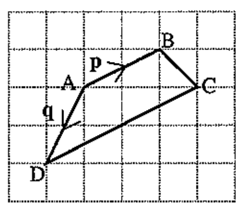
6. On a graph with origin at O, draw $\overrightarrow{OA} = \begin{pmatrix}5\\-1\end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix}6\\-7\end{pmatrix}$. Given that M is the mid-point of AB express the following as column vectors:
(a) $\overrightarrow{BA}$ (b) $\overrightarrow{BM}$ (c) $\overrightarrow{OM}$
Hence write down the coordinates of M.

7. On a graph with origin at O, draw $\overrightarrow{OA} = \begin{pmatrix}-2\\5\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}4\\2\end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix}-2\\-4\end{pmatrix}$.
(a) Given that M divides AB such that AM : MB = 2 : 1, express the following as column vectors:
(i) $\overrightarrow{BA}$ (ii) $\overrightarrow{BM}$ (iii) $\overrightarrow{OM}$
(b) Given that N divides AC such that AN : NC = 1 : 2, express the following as column vectors:
(i) $\overrightarrow{AC}$ (ii) $\overrightarrow{AN}$ (iii) $\overrightarrow{ON}$

8. In square ABCD, side AB has column vector $\begin{pmatrix}2\\1\end{pmatrix}$. Find two possible column vectors for $\overrightarrow{BC}$.

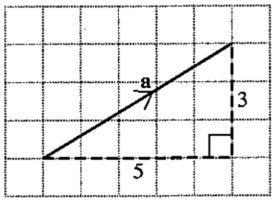
9. Rectangle KLMN has an area of 10 square units and $\overrightarrow{KL}$ has column vector $\begin{pmatrix}5\\0\end{pmatrix}$. Find two possible column vectors for $\overrightarrow{LM}$.

10. In the diagram, ABCD is a trapezium in which $\overrightarrow{DC} = 2\overrightarrow{AB}$.
If $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$ express in terms of **p** and **q**:
(a) $\overrightarrow{BD}$ (b) $\overrightarrow{AC}$ (c) $\overrightarrow{BC}$



11. Find the image of the vector $\begin{pmatrix}1\\3\end{pmatrix}$ after reflection in the following lines:
(a) $y = 0$ (b) $x = 0$ (c) $y = x$ (d) $y = -x$

Modulus of a vector



The modulus of a vector **a** is written |**a**| and represents the length (or magnitude) of the vector.

In the diagram above, $\mathbf{a} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

By Pythagoras' Theorem, $|\mathbf{a}| = \sqrt{(5^2 + 3^2)}$

$|\mathbf{a}| = \sqrt{34}$ units

In general if $\mathbf{x} = \begin{pmatrix} m \\ n \end{pmatrix}$, $|\mathbf{x}| = \sqrt{(m^2 + n^2)}$

Exercise 9

Questions **1** to **12** refer to the following vectors:

$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$\mathbf{e} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ $\mathbf{f} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

Find the following, leaving the answer in square root form where necessary.

1. $|\mathbf{a}|$ **2.** $|\mathbf{b}|$ **3.** $|\mathbf{c}|$ **4.** $|\mathbf{d}|$

5. $|\mathbf{e}|$ **6.** $|\mathbf{f}|$ **7.** $|\mathbf{a} + \mathbf{b}|$ **8.** $|\mathbf{c} - \mathbf{d}|$

9. $|2\mathbf{e}|$ **10.** $|\mathbf{f} + 2\mathbf{b}|$

11. (a) Find $|\mathbf{a} + \mathbf{c}|$. (b) Is $|\mathbf{a} + \mathbf{c}|$ equal to $|\mathbf{a}| + |\mathbf{c}|$?

12. (a) Find $|\mathbf{c} + \mathbf{d}|$. (b) Is $|\mathbf{c} + \mathbf{d}|$ equal to $|\mathbf{c}| + |\mathbf{d}|$?

13. If $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, find $|\overrightarrow{AC}|$.

14. If $\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, find $|\overrightarrow{PR}|$.

15. If $\overrightarrow{WX} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\overrightarrow{XY} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\overrightarrow{YZ} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $|\overrightarrow{WZ}|$.

16. Given that $\overrightarrow{OP} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} n \\ 3 \end{pmatrix}$, find:

(a) $|\overrightarrow{OP}|$ (b) a value for n if $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$

17. Given that $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 0 \\ m \end{pmatrix}$, find:

(a) $|\overrightarrow{OA}|$ (b) a value for m if $|\overrightarrow{OA}| = |\overrightarrow{OB}|$

18. Given that $\overrightarrow{LM} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{MN} = \begin{pmatrix} -15 \\ p \end{pmatrix}$, find:

(a) $|\overrightarrow{LM}|$ (b) a value for p if $|\overrightarrow{MN}| = 3|\overrightarrow{LM}|$

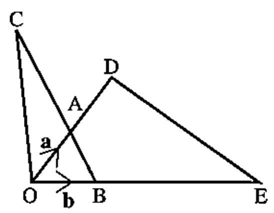
19. $\mathbf{a}$ and $\mathbf{b}$ are two vectors and $|\mathbf{a}| = 3$.
Find the value of $|\mathbf{a} + \mathbf{b}|$ when:
(a) $\mathbf{b} = 2\mathbf{a}$
(b) $\mathbf{b} = -3\mathbf{a}$
(c) $\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $|\mathbf{b}| = 4$

20. $\mathbf{r}$ and $\mathbf{s}$ are two vectors and $|\mathbf{r}| = 5$.
Find the value of $|\mathbf{r} + \mathbf{s}|$ when:
(a) $\mathbf{s} = 5\mathbf{r}$
(b) $\mathbf{s} = -2\mathbf{r}$
(c) $\mathbf{r}$ is perpendicular to $\mathbf{s}$ and $|\mathbf{s}| = 5$
(d) $\mathbf{s}$ is perpendicular to $(\mathbf{r} + \mathbf{s})$ and $|\mathbf{s}| = 3$

8.5 Vector geometry

Example

In the diagram, $\overrightarrow{OD} = 2\overrightarrow{OA}$, $\overrightarrow{OE} = 4\overrightarrow{OB}$, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
(a) Express $\overrightarrow{OD}$ and $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ respectively.
(b) Express $\overrightarrow{BA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(c) Express $\overrightarrow{ED}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(d) Given that $\overrightarrow{BC} = 3\overrightarrow{BA}$, express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(e) Express $\overrightarrow{EC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(f) Hence show that the points E, D and C lie on a straight line.



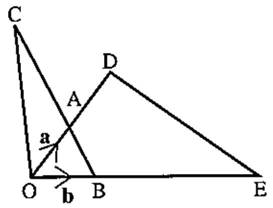
(a) $\overrightarrow{OD} = 2\mathbf{a}$
$\overrightarrow{OE} = 4\mathbf{b}$
(b) $\overrightarrow{BA} = -\mathbf{b} + \mathbf{a}$
(c) $\overrightarrow{ED} = -4\mathbf{b} + 2\mathbf{a}$
(d) $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$
$\overrightarrow{OC} = \mathbf{b} + 3(-\mathbf{b} + \mathbf{a})$
$\overrightarrow{OC} = -2\mathbf{b} + 3\mathbf{a}$
(e) $\overrightarrow{EC} = \overrightarrow{EO} + \overrightarrow{OC}$
$\overrightarrow{EC} = -4\mathbf{b} + (-2\mathbf{b} + 3\mathbf{a})$
$\overrightarrow{EC} = -6\mathbf{b} + 3\mathbf{a}$
(f) Using the results for $\overrightarrow{ED}$ and $\overrightarrow{EC}$, we see that $\overrightarrow{EC} = \frac{3}{2}\overrightarrow{ED}$.

$\overrightarrow{OD}$, $\overrightarrow{OC}$ and $\overrightarrow{OB}$ are called **position vectors**.
The position vector $\overrightarrow{OB}$ gives the position of B relative to the origin, O.

Since $\overrightarrow{EC}$ and $\overrightarrow{ED}$ are parallel vectors which both pass through the point E, the points E, D and C must lie on a straight line.

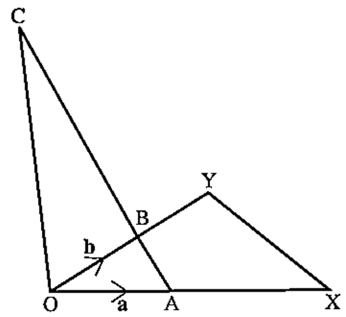
Exercise 10

1. $\overrightarrow{OD} = 2\overrightarrow{OA}$,
$\overrightarrow{OE} = 3\overrightarrow{OB}$,
$\overrightarrow{OA} = \mathbf{a}$ and
$\overrightarrow{OB} = \mathbf{b}$.



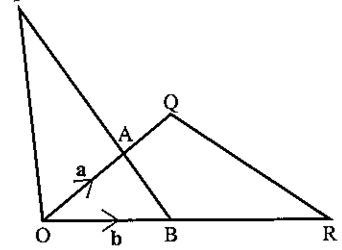
(a) Express $\overrightarrow{OD}$ and $\overrightarrow{OE}$ in terms of **a** and **b** respectively.
(b) Express $\overrightarrow{BA}$ in terms of **a** and **b**.
(c) Express $\overrightarrow{ED}$ in terms of **a** and **b**.
(d) Given that $\overrightarrow{BC} = 4\overrightarrow{BA}$, express $\overrightarrow{OC}$ in terms of **a** and **b**.
(e) Express $\overrightarrow{EC}$ in terms of **a** and **b**.
(f) Use the results for $\overrightarrow{ED}$ and $\overrightarrow{EC}$ to show that points E, D and C lie on a straight line.

2. $\overrightarrow{OY} = 2\overrightarrow{OB}$,
$\overrightarrow{OX} = \frac{5}{2}\overrightarrow{OA}$,
$\overrightarrow{OA} = \mathbf{a}$ and
$\overrightarrow{OB} = \mathbf{b}$.



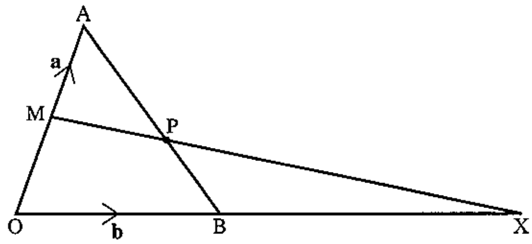
(a) Express $\overrightarrow{OY}$ and $\overrightarrow{OX}$ in terms of **b** and **a** respectively.
(b) Express $\overrightarrow{AB}$ in terms of **a** and **b**.
(c) Express $\overrightarrow{XY}$ in terms of **a** and **b**.
(d) Given that $\overrightarrow{AC} = 6\overrightarrow{AB}$, express $\overrightarrow{OC}$ in terms of **a** and **b**.
(e) Express $\overrightarrow{XC}$ in terms of **a** and **b**.
(f) Use the results for $\overrightarrow{XY}$ and $\overrightarrow{XC}$ to show that points X, Y and C lie on a straight line.

3. $\overrightarrow{OA} = \mathbf{a}$,
$\overrightarrow{OB} = \mathbf{b}$,
$\overrightarrow{AQ} = \frac{1}{2}\mathbf{a}$,
$\overrightarrow{BR} = \mathbf{b}$ and
$\overrightarrow{AP} = 2\overrightarrow{BA}$.



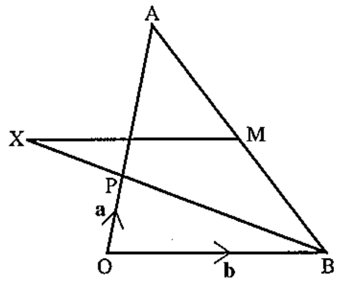
(a) Express $\overrightarrow{BA}$ and $\overrightarrow{BP}$ in terms of **a** and **b**.
(b) Express $\overrightarrow{RQ}$ in terms of **a** and **b**.
(c) Express $\overrightarrow{QA}$ and $\overrightarrow{QP}$ in terms of **a** and **b**.
(d) Using the vectors for $\overrightarrow{RQ}$ and $\overrightarrow{QP}$, show that R, Q and P lie on a straight line.

4. In the diagram, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, M is the mid-point of OA and P lies on AB such that $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$.



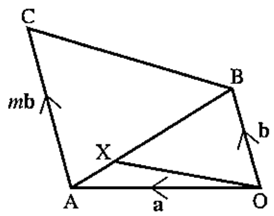
(a) Express $\overrightarrow{AB}$ and $\overrightarrow{AP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Express $\overrightarrow{MA}$ and $\overrightarrow{MP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(c) If X lies on OB produced such that OB = BX, express $\overrightarrow{MX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(d) Show that MPX is a straight line.

5. $\overrightarrow{OP} = \mathbf{a}$,
$\overrightarrow{OA} = 3\mathbf{a}$,
$\overrightarrow{OB} = \mathbf{b}$ and
M is the mid-point of AB.



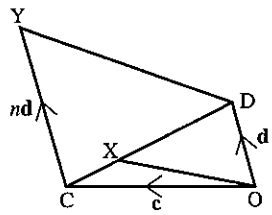
(a) Express $\overrightarrow{BP}$ and $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Express $\overrightarrow{MB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(c) If X lies on BP produced so that $\overrightarrow{BX} = k.\ \overrightarrow{BP}$, express $\overrightarrow{MX}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and k.
(d) Find the value of k if MX is parallel to BO.

6. AC is parallel to OB,
$\overrightarrow{AX} = \frac{1}{4}\overrightarrow{AB}$,
$\overrightarrow{OA} = \mathbf{a}$,
$\overrightarrow{OB} = \mathbf{b}$ and
$\overrightarrow{AC} = m\mathbf{b}$.



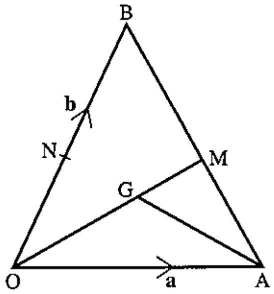
(a) Express $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Express $\overrightarrow{AX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(c) Express $\overrightarrow{BC}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and m.
(d) Given that OX is parallel to BC, find the value of m.

7. CY is parallel to OD,
$\overrightarrow{CX} = \frac{1}{5}\overrightarrow{CD}$,
$\overrightarrow{OC} = \mathbf{c}$,
$\overrightarrow{OD} = \mathbf{d}$ and
$\overrightarrow{CY} = n\mathbf{d}$.



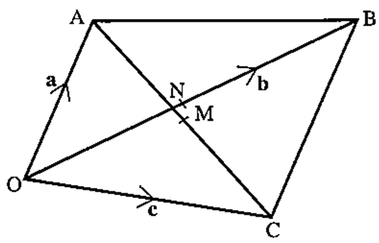
(a) Express $\overrightarrow{CD}$ in terms of $\mathbf{c}$ and $\mathbf{d}$.
(b) Express $\overrightarrow{CX}$ in terms of $\mathbf{c}$ and $\mathbf{d}$.
(c) Express $\overrightarrow{OX}$ in terms of $\mathbf{c}$ and $\mathbf{d}$.
(d) Express $\overrightarrow{DY}$ in terms of $\mathbf{c}$, $\mathbf{d}$ and n.
(e) Given that OX is parallel to DY, find the value of n.

8. M is the mid-point of AB,
N is the mid-point of OB,
$\overrightarrow{OA} = \mathbf{a}$ and
$\overrightarrow{OB} = \mathbf{b}$.



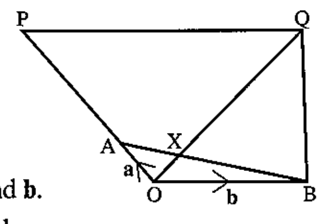
(a) Express $\overrightarrow{AB}$, $\overrightarrow{AM}$ and $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Given that G lies on OM such that
OG : GM = 2 : 1, express $\overrightarrow{OG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(c) Express $\overrightarrow{AG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(d) Express $\overrightarrow{AN}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(e) Show that $\overrightarrow{AG} = m\overrightarrow{AN}$ and find the value of m.

9. M is the mid-point of AC and N is the mid-point of OB,
$\overrightarrow{OA} = \mathbf{a}$,
$\overrightarrow{OB} = \mathbf{b}$ and
$\overrightarrow{OC} = \mathbf{c}$.



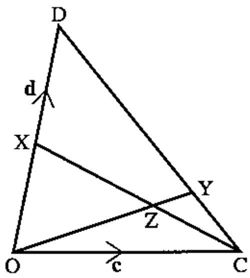
(a) Express $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Express $\overrightarrow{ON}$ in terms of $\mathbf{b}$.
(c) Express $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.
(d) Express $\overrightarrow{AM}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.
(e) Express $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.
(f) Express $\overrightarrow{NM}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
(g) If N and M coincide, write down an equation connecting $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

10. $\overrightarrow{OA} = \mathbf{a}$ and
$\overrightarrow{OB} = \mathbf{b}$.



(a) Express $\overrightarrow{BA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(b) Given that $\overrightarrow{BX} = m\overrightarrow{BA}$, show that $\overrightarrow{OX} = m\mathbf{a} + (1 - m)\mathbf{b}$.
(c) Given that OP = 4**a** and $\overrightarrow{PQ} = 2\mathbf{b}$, express $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(d) Given that $\overrightarrow{OX} = n\overrightarrow{OQ}$ use the results for $\overrightarrow{OX}$ and $\overrightarrow{OQ}$ to find the values of m and n.

11. X is the mid-point of OD, Y lies on CD such that
$\overrightarrow{CY} = \frac{1}{4}\overrightarrow{CD}$,
$\overrightarrow{OC} = \mathbf{c}$ and
$\overrightarrow{OD} = \mathbf{d}$.



(a) Express $\overrightarrow{CD}$, $\overrightarrow{CY}$ and $\overrightarrow{OY}$ in terms of $\mathbf{c}$ and $\mathbf{d}$.
(b) Express $\overrightarrow{CX}$ in terms of $\mathbf{c}$ and $\mathbf{d}$.
(c) Given that $\overrightarrow{CZ} = h\overrightarrow{CX}$, express $\overrightarrow{OZ}$ in terms of $\mathbf{c}$, $\mathbf{d}$ and h.
(d) If $\overrightarrow{OZ} = k\overrightarrow{OY}$, form an equation and hence find the values of h and k.

8.6 Functions

The idea of a function is used in almost every branch of mathematics.
The two common notations used are:
(a) $f(x) = x^2 + 4$ (b) $f: x \mapsto x^2 + 4$

We may interpret (b) as follows: 'function f such that x is mapped onto $x^2 + 4$'.

Example

If $f(x) = 3x - 1$ and $g(x) = 1 - x^2$ find:

(a) $f(2)$ (b) $f(-2)$ (c) $g(0)$ (d) $g(3)$ (e) x if $f(x) = 1$

(a) $f(2) = 5$ (b) $f(-2) = -7$ (c) $g(0) = 1$ (d) $g(3) = -8$
(e) If $f(x) = 1$
Then $3x - 1 = 1$
$3x = 2$
$x = \frac{2}{3}$

Flow diagrams

The function f in the example consisted of two simpler functions as illustrated by a flow diagram.

$$x \longrightarrow \boxed{\text{Multiply by 3}} \xrightarrow{3x} \boxed{\text{Subtract 1}} \xrightarrow{3x-1}$$

It is obviously important to 'multiply by 3' and 'subtract 1' in the correct order.

Example

Draw flow diagrams for the functions:

(a) $f: x \mapsto (2x + 5)^2$, (b) $g(x) = \dfrac{5 - 7x}{3}$

(a) $\xrightarrow{x} \boxed{\text{Multiply by 2}} \xrightarrow{2x} \boxed{\text{add 5}} \xrightarrow{2x+5} \boxed{\text{square}} \xrightarrow{(2x+5)^2}$

(b) $\xrightarrow{x} \boxed{\text{Multiply by }(-7)} \xrightarrow{-7x} \boxed{\text{add 5}} \xrightarrow{5-7x} \boxed{\text{divide by 3}} \xrightarrow{\frac{5-7x}{3}}$

Exercise 11

1. Given the functions $h: x \mapsto x^2 + 1$ and $g: x \mapsto 10x + 1$. Find:
(a) $h(2)$, $h(-3)$, $h(0)$ (b) $g(2)$, $g(10)$, $g(-3)$

In questions **2** to **15**, draw a flow diagram for each function.

2. $f: x \mapsto 5x + 4$
3. $f: x \mapsto 3(x - 4)$
4. $f: x \mapsto (2x + 7)^2$
5. $f: x \mapsto \left(\dfrac{9 + 5x}{4}\right)$
6. $f: x \mapsto \dfrac{4 - 3x}{5}$
7. $f: x \mapsto 2x^2 + 1$
8. $f: x \mapsto \dfrac{3x^2}{2} + 5$
9. $f: x \mapsto \sqrt{(4x - 5)}$
10. $f: x \mapsto 4\sqrt{(x^2 + 10)}$

11. $f: x \mapsto (7-3x)^2$ **12.** $f: x \mapsto 4(3x+1)^2+5$ **13.** $f: x \mapsto 5-x^2$

14. $f: x \mapsto \dfrac{10\sqrt{(x^2+1)}+6}{4}$ **15.** $f: x \mapsto \left(\dfrac{x^3}{4}+1\right)^2 -6$

For questions **16**, **17** and **18**, the functions f, g and h are defined as follows:

$f: x \mapsto 1-2x$

$g: x \mapsto \dfrac{x^3}{10}$

$h: x \mapsto \dfrac{12}{x}$

16. Find:
(a) $f(5)$, $f(-5)$, $f(\frac{1}{4})$
(b) $g(2)$, $g(-3)$, $g(\frac{1}{2})$
(c) $h(3)$, $h(10)$, $h(\frac{1}{3})$

17. Find:
(a) x if $f(x)=1$ (b) x if $f(x)=-11$ (c) x if $h(x)=1$

18. Find:
(a) y if $g(y)=100$ (b) z if $h(z)=24$ (c) w if $g(w)=0{\cdot}8$

For questions **19** and **20**, the functions k, l and m are defined as follows:

$k: x \mapsto \dfrac{2x^2}{3}$

$l: x \mapsto \sqrt{[(y-1)(y-2)]}$

$m: x \mapsto 10-x^2$

19. Find:
(a) $k(3)$, $k(6)$, $k(-3)$
(b) $l(2)$, $l(0)$, $l(4)$
(c) $m(4)$, $m(-2)$, $m(\frac{1}{2})$

20. Find:
(a) x if $k(x)=6$ (b) x if $m(x)=1$
(c) y if $k(y)=2\frac{2}{3}$ (d) p if $m(p)=-26$

21. $f(x)$ is defined as the product of the digits of x,
e.g. $f(12)=1\times 2=2$
(a) Find: (i) $f(25)$ (ii) $f(713)$
(b) If x is an integer with three digits, find:
(i) x such that $f(x)=1$
(ii) the largest x such that $f(x)=4$
(iii) the largest x such that $f(x)=0$
(iv) the smallest x such that $f(x)=2$

22. $g(x)$ is defined as the sum of the prime factors of x, e.g. $g(12) = 2 + 3 = 5$. Find:
(a) $g(10)$ (b) $g(21)$ (c) $g(36)$
(d) $g(99)$ (e) $g(100)$ (f) $g(1000)$

23. $h(x)$ is defined as the number of letters in the English word describing the number x, e.g. $h(1) = 3$. Find:
(a) $h(2)$ (b) $h(11)$ (c) $h(18)$
(d) the largest value of x for which $h(x) = 3$

24. If $f: x \mapsto$ next prime number greater than x, find:
(a) $f(7)$ (b) $f(14)$ (c) $f[f(3)]$

25. If $g: x \to 2^x + 1$, find:
(a) $g(2)$ (b) $g(4)$ (c) $g(-1)$
(d) the value of x if $g(x) = 9$

26. The function f is defined as $f: x \to ax + b$ where a and b are constants.
If $f(1) = 8$ and $f(4) = 17$, find the values of a and b.

27. The function g is defined as $g(x) = ax^2 + b$ where a and b are constants.
If $g(2) = 3$ and $g(-3) = 13$, find the values of a and b.

28. Functions h and k are defined as follows:
$h: x \mapsto x^2 + 1$, $k: x \mapsto ax + b$, where a and b are constants.
If $h(0) = k(0)$ and $k(2) = 15$, find the values of a and b.

Composite functions

The function $f: x \mapsto 3x + 2$ is itself a composite function, consisting of two simpler functions: 'multiply by 3' and 'add 2'.

If $f: x \mapsto 3x + 2$ and $g: x \mapsto x^2$ then fg is a composite function where g is performed first and then f is performed on the result of g.

The function fg may be found using a flow diagram.

$$\xrightarrow{x} \underset{g}{\boxed{\text{square}}} \xrightarrow{x^2} \underset{f}{\boxed{\text{multiply by 3}}} \xrightarrow{3x^2} \boxed{\text{add 2}} \xrightarrow{3x^2 + 2}$$

Thus $fg: x \mapsto 3x^2 + 2$

Inverse functions

If a function f maps a number n onto m, then the inverse function f^{-1} maps m onto n. The inverse of a given function is found using a flow diagram.

Example

Find the inverse of f where $f: x \to \dfrac{5x-2}{3}$.

(a) Draw a flow diagram for f.

$$\xrightarrow{x} \boxed{\text{Multiply by 5}} \xrightarrow{5x} \boxed{\text{subtract 2}} \xrightarrow{5x-2} \boxed{\text{divide by 3}} \xrightarrow{\frac{5x-2}{3}}$$

(b) Draw a new flow diagram with each operation replaced by its inverse. Start with x on the right.

$$\xleftarrow{\frac{3x+2}{5}} \boxed{\text{divide by 5}} \xleftarrow{3x+2} \boxed{\text{add 2}} \xleftarrow{3x} \boxed{\text{Multiply by 3}} \xleftarrow{x}$$

Thus the inverse of f is given by

$$f^{-1}: x \mapsto \frac{3x+2}{5} \quad \text{or} \quad f^{-1}(x) = \frac{3x+2}{5}$$

Exercise 12

For questions **1** and **2**, the functions f, g and h are as follows:

$f: x \mapsto 4x$
$g: x \mapsto x+5$
$h: x \mapsto x^2$

1. Find the following in the form '$x \mapsto \ldots$'
(a) fg (b) gf (c) hf (d) fh
(e) gh (f) fgh (g) hfg

2. Find:
(a) x if $hg(x) = h(x)$ (b) x if $fh(x) = gh(x)$

For questions **3**, **4** and **5**, the functions f, g and h are as follows:

$f: x \mapsto 2x$
$g: x \mapsto x-3$
$h: x \mapsto x^2$

3. Find the following in the form '$x \mapsto \ldots$'
(a) fg (b) gf (c) gh
(d) hf (e) ghf (f) hgf

4. Evaluate:
(a) $fg(4)$ (b) $gf(7)$ (c) $gh(-3)$
(d) $fgf(2)$ (e) $ggg(10)$ (f) $hfh(-2)$

5. Find:
(a) x if $f(x) = g(x)$ (b) x if $hg(x) = gh(x)$
(c) x if $gf(x) = 0$ (d) x if $fg(x) = 4$

For questions **6**, **7** and **8**, the functions, l, m and n are as follows:

$l : x \mapsto 2x + 1$
$m : x \mapsto 3x - 1$
$n : x \mapsto x^2$

6. Find the following in the form '$x \mapsto \ldots$'

(a) lm (b) ml (c) ln
(d) nm (e) lnm (f) mln

7. Find:

(a) $lm(2)$ (b) $nl(1)$ (c) $mn(-2)$
(d) $mm(2)$ (e) $nln(2)$ (f) $llm(0)$

8. Find:

(a) x if $l(x) = m(x)$
(b) two values of x if $nl(x) = nm(x)$
(c) x if $ln(x) = mn(x)$

In questions **9** to **22**, find the inverse of each function in the form '$x \mapsto \ldots$'

9. $f : x \mapsto 5x - 2$

10. $f : x \mapsto 5(x - 2)$

11. $f : x \mapsto 3(2x + 4)$

12. $g : x \mapsto \dfrac{2x + 1}{3}$

13. $f : x \mapsto \dfrac{3(x - 1)}{4}$

14. $g : x \mapsto 2(3x + 4) - 6$

15. $h : x \mapsto \frac{1}{2}(4 + 5x) + 10$

16. $k : x \mapsto -7x + 3$

17. $j : x \mapsto \dfrac{12 - 5x}{3}$

18. $l : x \mapsto \dfrac{4 - x}{3} + 2$

19. $m : x \mapsto \dfrac{\left[\dfrac{(2x - 1)}{4} - 3\right]}{5}$

20. $f : x \mapsto \dfrac{3(10 - 2x)}{7}$

21. $g : x \mapsto \left[\dfrac{\dfrac{x}{4} + 6}{5}\right] + 7$

22. A calculator has the following function buttons:

$x \mapsto x^2$; $x \mapsto \sqrt{x}$; $x \mapsto \frac{1}{x}$; $x \mapsto \log x$;
$x \mapsto \ln x$; $x \mapsto \sin x$; $x \mapsto \cos x$; $x \mapsto \tan x$; $x \mapsto x!$

x! is 'x factorial'
4! = 4 × 3 × 2 × 1
3! = 3 × 2 × 1 etc

Find which button was used for the following input/outputs:

(a) $1\,000\,000 \to 1000$ (b) $1000 \to 3$
(c) $3 \to 6$ (d) $0{\cdot}2 \to 0{\cdot}04$
(e) $10 \to 0{\cdot}1$ (f) $45 \to 1$
(g) $0{\cdot}5 \to 2$ (h) $64 \to 8$
(i) $60 \to 0{\cdot}5$ (j) $1 \to 0$
(k) $135 \to -1$ (l) $10 \to 3\,628\,800$
(m) $0 \to 1$ (n) $30 \to 0{\cdot}5$
(o) $90 \to 0$ (p) $0{\cdot}4 \to 2{\cdot}5$
(q) $4 \to 24$ (r) $1\,000\,000 \to 6$

Revision exercise 8A

1. Given that $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8\}$,
   $A = \{1, 3, 5\}$, $B = \{5, 6, 7\}$, list the members of the sets:
   (a) $A \cap B$ (b) $A \cup B$ (c) A'
   (d) $A' \cap B'$ (e) $A \cup B'$

2. The sets P and Q are such that $n(P \cup Q) = 50$, $n(P \cap Q) = 9$ and $n(P) = 27$. Find the value of $n(Q)$.

3. Draw three diagrams similar to Figure 1, and shade the following
   (a) $Q \cap R'$ (b) $(P \cup Q) \cap R$ (c) $(P \cap Q) \cap R'$

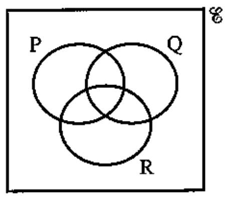


Figure 1

4. Describe the shaded regions in Figures 2 and 3.

(a)

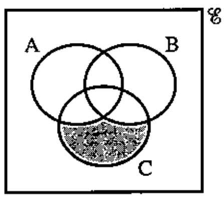


Figure 2

(b)

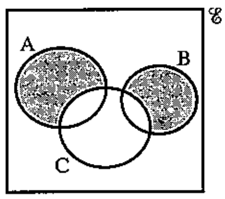
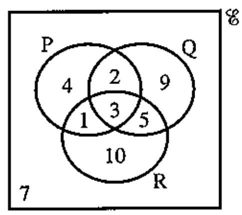


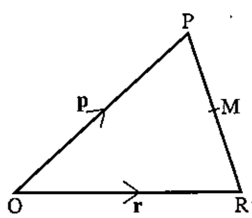
Figure 3

5. Given that $\mathscr{E}$ = {people on a train}, M = {males},
   T = {people over 25 years old} and S = {snooker players},
   (a) express in set notation:
      (i) all the snooker players are over 25
      (ii) some snooker players are women
   (b) express in words: $T \cap M' = \varnothing$

6. The figures in the diagram indicate the number of elements in each subset of $\mathscr{E}$.
   (a) Find $n(P \cap R)$.
   (b) Find $n(Q \cup R)'$.
   (c) Find $n(P' \cap Q')$.



7. In $\triangle OPR$, the mid-point of PR is M.
   If $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OR} = \mathbf{r}$, find in terms of $\mathbf{p}$ and $\mathbf{r}$:
   (a) $\overrightarrow{PR}$ (b) $\overrightarrow{PM}$ (c) $\overrightarrow{OM}$



8. If $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, find:
(a) $|\mathbf{b}|$ (b) $|\mathbf{a}+\mathbf{b}|$ (c) $|2\mathbf{a}-\mathbf{b}|$

9. If $4\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 1 \\ m \end{pmatrix} = 3\begin{pmatrix} n \\ -6 \end{pmatrix}$, find the values of m and n.

10. The points O, A and B have coordinates (0, 0), (5, 0) and (−1, 4) respectively. Write as column vectors.
(a) $\overrightarrow{OB}$ (b) $\overrightarrow{OA}+\overrightarrow{OB}$ (c) $\overrightarrow{OA}-\overrightarrow{OB}$
(d) $\overrightarrow{OM}$ where M is the mid-point of AB.

11. In the parallelogram OABC, M is the mid-point of AB and N is the mid-point of BC.
If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$, express in terms of **a** and **c**:
(a) $\overrightarrow{CA}$ (b) $\overrightarrow{ON}$ (c) $\overrightarrow{NM}$
Describe the relationship between CA and NM.

12. The vectors **a**, **b**, **c** are given by:
$\mathbf{a} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -1 \\ 17 \end{pmatrix}$
Find numbers m and n so that $m\mathbf{a} + n\mathbf{b} = \mathbf{c}$.

13. Given that $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ and that M is the mid-point of PQ, express as column vectors:
(a) $\overrightarrow{PQ}$ (b) $\overrightarrow{PM}$ (c) $\overrightarrow{OM}$

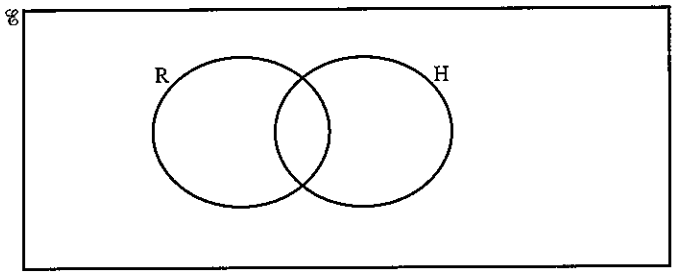
14. Given $f: x \mapsto 2x - 3$ and $g: x \mapsto x^2 - 1$, find:
(a) $f(-1)$ (b) $g(-1)$ (c) $fg(-1)$ (d) $gf(3)$
Write the function ff in the form '$ff: x \mapsto \ldots$'

15. If $f: x \mapsto 3x + 4$ and $h: x \mapsto \dfrac{x-2}{5}$
express f^{-1} and h^{-1} in the form '$x \mapsto \ldots$'.
Find: (a) $f^{-1}(13)$ (b) the value of z if $f(z) = 20$

16. Given that $f(x) = x - 5$, find:
(a) the value of s such that $f(s) = -2$
(b) the values of t such that $t \times f(t) = 0$

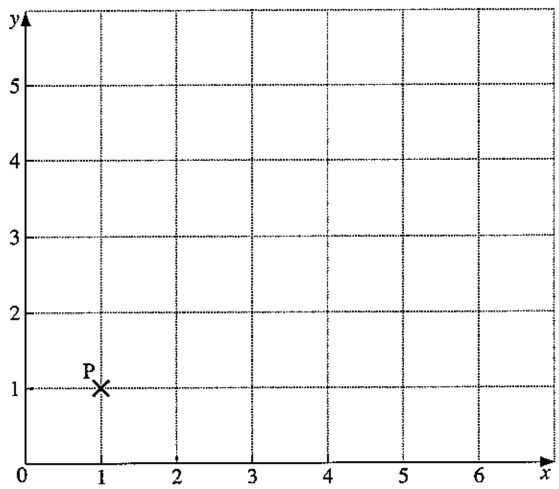
Examination exercise 8B

1. $\mathscr{E} = \{x : x \text{ is a positive integer}\}$, $A = \{x : 3x - 2 < 15\}$,
   $B = \{x : 4x + 1 \geqslant 13\}$.
   (a) Find $n(A)$. (b) List the set $A \cap B$. J 98 2

2. 

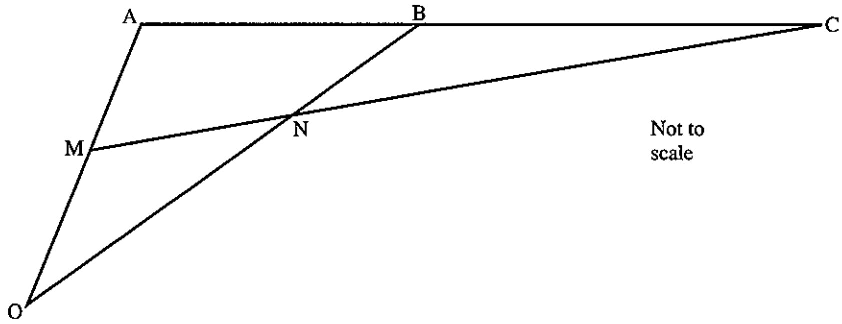
   $\mathscr{E} = \{\text{quadrilaterals}\}$, $R = \{\text{rectangles}\}$ and $H = \{\text{rhombuses}\}$.
   (a) Which special quadrilaterals belong to $R \cap H$?
   (b) $P = \{\text{parallelograms}\}$. Draw and label P on the diagram above.
   (c) A quadrilateral, x, has unequal diagonals which bisect each other at 90°.
   Mark x on the diagram above. J 95 2

3. $\mathscr{E} = \{x : x \text{ is an integer and } 2 \leqslant x \leqslant 10\}$,
   $A = \{\text{multiples of 3}\}$,
   $B = \{\text{prime numbers}\}$.
   List the members of:
   (a) $A \cap B$ (b) $A \cup B$ (c) $(A \cup B)'$ N 95 2

4. 

   P is the point (1, 1). The vector $\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
   (a) Find the vector $\mathbf{m} + 2\mathbf{n}$.
   (b) $\overrightarrow{PQ} = \mathbf{m} + 2\mathbf{n}$. Find the position vector of Q.
   (c) Calculate $|\mathbf{m}|$, the magnitude of $\mathbf{m}$. J 96 2

5. In the diagram, O is the origin, ABC is a straight line and M is the mid-point of OA.



$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{AC} = 3\overrightarrow{AB}$

(a) Find, in terms of **a** and/or **b**, in their simplest forms:
(i) $\overrightarrow{MA}$ (ii) $\overrightarrow{AB}$ (iii) $\overrightarrow{AC}$ (iv) $\overrightarrow{MC}$
(v) the position vector of C.

(b) It is also given that $\overrightarrow{MN} = \frac{1}{5}\overrightarrow{MC}$.
(i) Find $\overrightarrow{ON}$ in terms of **a** and/or **b**.
(ii) Find the ratio ON : NB. N 95 4

6. $f(x) = \sqrt{3x+1}$ for $x \geqslant -\frac{1}{3}$
(a) Find $f(3\frac{3}{4})$.
(b) Solve $f(x) = 5$.
(c) Find $f^{-1}(x)$. J 95 2

7. $f(x) = x^2 - 16$ and $g(x) = 5x + 2$ for all values of x.
(a) Find:
(i) $f(10)$
(ii) $f(-2)$
(b) Find $g^{-1}(x)$, the inverse of $g(x)$.
(c) Find $fg(x)$, giving the answer in its simplest terms.
(d) Find the two values of x for which $f(x) = g(x)$.
Give your answers correct to two decimal places. N 98 4

8. $f(x) = 3x^2 - 3x + 1$
(a) Find the exact value of $f(\frac{1}{6})$.
(b) $f(1-x) = 3(1-x)^2 - 3(1-x) + 1$
Show that $f(1-x) = f(x)$.
(c) Write down the value of $f(\frac{5}{6})$. N 96 2

9. $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Find $3\mathbf{a} - 2\mathbf{b}$. J 03 2